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On the existence of supergravity duals to D1-D5 CFT states

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Abstract: We define a metric operator in the $\frac{1}{2}$-BPS sector of the D1-D5 CFT, the eigenstates of which have a good semi-classical supergravity dual; the non-eigenstates cannot be mapped to semi-classical gravity duals. We also analyse how the data defining a CFT state manifests itself in the gravity side, and show that it is arranged into a set of multipoles. Interestingly, we find that quantum mechanical interference in the CFT can have observable manifestations in the semi-classical gravity dual. We also point out that the multipoles associated to the normal statistical ensemble fluctuate wildly, indicating that the mixed thermal state should not be associated to a semi-classical geometry.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Gauge-gravity correspondence.

Contents

1. Introduction

Recently there has been considerable progress in using the AdS/CFT correspondence to understand quantum gravity, especially in the form of explicit mappings from certain CFT's to their dual semi-classical geometries. The first such system was the set of LLM geometries: an explicit map from states of $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ SU(N) SYM to their dual supergravity solutions [1]. This map was further developed and analysed among others in [2 − 7]. Progress in extending this mapping to the $\frac{1}{4}$- and $\frac{1}{8}$-BPS sectors has been made in [8 − 12]. Another such map was proposed between the $\frac{1}{2}$-BPS sectors of the D1-D5 black hole and its dual field theory in [13], and also in [14 − 16]; while similar mappings were introduced and analysed for the set of Lin-Maldacena geometries in [17, 18]. All of these mappings lend support to the proposal that gravity is thermodynamic in nature.

In all cases the supergravity analyses were formulated in terms of classical solutions, but any such mapping must also extend to the quantum level. Such an extension for the

LLM system was proposed in [4], and in this note we apply the methods developed in that paper to the D1-D5 system.

We propose a 'metric' operator in the CFT: an operator whose eigenstates are dual to semi-classical geometries via the mapping given in [13]. The states that fail to be eigenstates, however, cannot be mapped to spacetimes with unique metrics.

We also analyse how the data characterising the field theory state shows up in the asymptotic form of the spacetime metric. We find the data to be arranged into a set of multipoles, the first of which was already considered in [13, 19] as the dipole operator. We also find that certain terms in the metric only show up if the CFT dual state is a superposition of basis states, and demonstrate the measurability of these interference effects. Both of these results are highly analogous to what was found for the LLM geometries in [3, 4].

Finally, we point out that the thermal ensemble, consisting of a sum over all states with the total twist N fixed using a lagrange multiplier β, is not an eigenstate of the metric operator due to the large fluctuations inherent in the ensemble. This is again highly analogous to what was found for the LLM case in [6], but we show that the method used there to restrict the ensemble is incapable of sufficiently constraining the ensemble in the D1-D5 case.

The paper is structured as follows. In section 2 we present a brief review of the D1-D5 system and the map proposed in [13]. In section 3 we construct the asymptotic expansion of the metric and find a set of multipoles. In section 4 we proceed to use these multipoles to motivate our definition of the metric operator, and define the approximate eigenstates of this operator. In section 5 we consider a more general asymptotic expansion of the metric and find the terms due to interference between basis states. In section 6 we consider the thermal ensemble, and we conclude in section 7 with some comments.

2. Review

We begin by briefly reviewing the D1-D5 system; for a more comprehensive review the reader is referred to [20 − 26]. The D1-D5 CFT, which is dual to type IIB string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$, is a marginal deformation of the $(1+1)$-dimensional orbifold sigma model with target space

$$\mathcal{M}_0 = \left(\mathrm{T}^4\right)^N / \mathrm{S}_N, \tag{2.1}$$

where N is related to the AdS scale and S_N is the permutation group. This duality arises as the decoupling limit of type IIB string theory on $\mathrm{M}^{1,4} \times \mathrm{S}^1 \times \mathrm{T}^4$ with N_1 D1-branes wrapping the S^1 and N_5 D5-branes wrapping $\mathrm{S}^1 \times \mathrm{T}^4$, where the parameters are related by $N = N_1 N_5$.

Gravity solutions: the microstate geometries of the D1-D5 system are well known and can be written as

$$ds^2 = \frac{1}{\sqrt{f_1 f_5}}\left[-(dt+A)^2 + (dy+B)^2\right] + \sqrt{f_1 f_5}\, d\vec{x}^2 + \sqrt{\frac{f_1}{f_5}}\, d\vec{z}^2, \tag{2.2}$$

$$e^{2\Phi} = \frac{f_1}{f_5}, \tag{2.3}$$

$$C = \frac{1}{f_1}(dt + A) \wedge (dy + B) + \mathcal{C}, \tag{2.4}$$

$$dB = *_4 dA, \tag{2.5}$$

$$d\mathcal{C} = - *_4 df_5, \tag{2.6}$$

$$f_5 = 1 + \frac{Q_5}{L} \int_0^L \frac{ds}{|\vec{x} - \vec{F}(s)|^2}, \tag{2.7}$$

$$f_1 = 1 + \frac{Q_5}{L} \int_0^L \frac{|\vec{F}'(s)|^2 ds}{|\vec{x} - \vec{F}(s)|^2}, \tag{2.8}$$

$$A_i = \frac{Q_5}{L} \int_0^L \frac{F_i'(s) ds}{|\vec{x} - \vec{F}(s)|^2}. \tag{2.9}$$

Here y and $\vec{z}$ parametrize the S^1 and T^4 respectively. The coordinate radius of the S^1 is R, while the coordinate volume of the T^4 is V_4. The charges Q_1 and Q_5 are related to N_1 and N_5 by

$$Q_5 = g_s N_5, \quad Q_1 = \frac{g_s}{V_4} N_1. \tag{2.10}$$

All these solutions are parametrized in terms of a closed curve $\vec{F}(s)$ in $\mathbb{R}^4$, which we expand as a Fourier series as

$$\vec{F}(s) = \mu \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{\sqrt{2|k|}} \vec{d}_k e^{i \frac{2\pi k}{L} s}, \tag{2.11}$$

where s ranges from 0 to L and $\vec{d}_k = (d_k^1, d_k^2, d_k^3, d_k^4) = \vec{d}_{-k}^*$. Note that the fermionic oscillations as well as oscillations on the T^4 have been omitted, as we are only interested in fluctuations in the $\mathbb{R}^4$. Additionally,

$$\mu = \frac{g_s}{R\sqrt{V_4}}. \tag{2.12}$$

The parameter L satisfies

$$LR = 2\pi Q_5, \tag{2.13}$$

and due to fixed length of the original string there is an additional constraint

$$Q_1 = \frac{Q_5}{L} \int_0^L |\vec{F}'(s)|^2 ds. \tag{2.14}$$

It was shown in [27] that the space of classical solutions can be quantized to yield a finite number of quantum states. The quantized system is given by[1]

$$\left[d_k^a, d_l^b\right] = \delta^{ab} \delta_{kl}, \tag{2.15}$$

$$\langle \int_0^L : |\vec{F}'(s)|^2 : ds \rangle = \frac{(2\pi)^2 \mu^2 N}{L}, \tag{2.16}$$

$$N = N_1 N_5 = \sum_{k=1}^{\infty} k \langle \vec{d}_k^\dagger \cdot \vec{d}_k \rangle. \tag{2.17}$$

[1]This is the only time we use $\vec{d}_k$'s as operators. Our notation is such that d_k's are complex numbers, while c_k, $c_k^\dagger$ denote annihilation and creation operators.

Field theory states: the Ramond ground states of the CFT are in one to one correspondence with states at level N of a Fock space of a system composed of 8 bosonic and 8 fermionic oscillators. We shall retain only four of these oscillators; the bosonic ones that correspond to fluctuations in the transverse $\mathbb{R}^4$. Thus a basis for the states can be written as

$$|\{N_k\}\rangle = \prod_{k=1}^{\infty}\prod_{a=1}^{4}\frac{1}{\sqrt{N_k^a!}}\left(c_k^{a\dagger}\right)^{N_k^a}|0\rangle, \quad \text{with} \sum_{k=1}^{\infty}\sum_{a=1}^{4} kN_k^a = N. \tag{2.18}$$

For convenience we shall write $\vec{c}_k^{\,\dagger} \equiv \vec{c}_{-k}$ for positive k, so that the notation $\vec{c}$ includes both the creation and annihilation operators. It was proposed in [13] to associate a phase space density $f(\vec{d})$ to each state $|\psi\rangle$ by

$$f_\psi(\vec{d}) = \frac{\langle 0|e^{\sum_{k=1}^{\infty}\vec{d}_k\cdot\vec{c}_k}|\psi\rangle\langle\psi|e^{\sum_{k=1}^{\infty}\vec{d}_k^*\cdot\vec{c}_k^{\dagger}}|0\rangle}{\langle 0|e^{\sum_{k=1}^{\infty}\vec{d}_k\cdot\vec{c}_k}e^{\sum_{k=1}^{\infty}\vec{d}_k^*\cdot\vec{c}_k^{\dagger}}|0\rangle}. \tag{2.19}$$

It can be shown that this distribution function corresponds to anti-normal ordering prescription in the quantum system, and can be used to compute expectation values of anti-normal ordered operators as

$$\int_{\vec{d}} f_\psi(\vec{d})\, g(\vec{d}) = \langle\psi|:g(\vec{c}):_A|\psi\rangle. \tag{2.20}$$

Also, the distribution corresponding to the basis state (2.18) can be easily computed and gives

$$f_{\{N_k^a\}}(\vec{d}) = \prod_{k=1}^{\infty}\prod_{a=1}^{4} e^{-d_k^a d_{-k}^a}\frac{\left(d_k^a d_{-k}^a\right)^{N_k^a}}{N_k^a!}. \tag{2.21}$$

In addition to this basis, we will often find it useful to work with coherent states. These can be defined as

$$|\{\vec{\tilde{d}}\}\rangle = e^{-\frac{\vec{\tilde{d}}_k\cdot\vec{\tilde{d}}_k^*}{2}}\mathrm{P}_N e^{\vec{\tilde{d}}_k^*\cdot\vec{c}_k^{\dagger}}|0\rangle, \tag{2.22}$$

where $\vec{\tilde{d}}_k \in \mathbb{C}^4$ for all k, and P_N is a projection operator to the twist N subspace of the Fock space. Note that we are suppressing the sums over k in the exponents, and that this definition differs from the definition in [13] by a normalization factor. With this definition one finds the corresponding distribution to be

$$f_{\vec{\tilde{d}}}(\vec{d}) = \prod_{k=1}^{\infty} e^{-|\vec{d}_k-\vec{\tilde{d}}_k|^2} + \mathcal{O}(\frac{1}{N}), \tag{2.23}$$

where the subleading correction arises because of the projection operator P_N, and will vanish in the $N\to\infty$ limit.

Using this distribution, it was proposed in [13] that the microstate geometry dual to state $|\Psi\rangle$ should be given by

$$f_5 = 1 + \frac{Q_5}{L}\mathcal{N}\int_{\vec{d}}\int_0^L \frac{f_\Psi(\vec{d})\,ds}{|\vec{x}-\vec{F}(s)|^2}, \tag{2.24}$$

$$f_1 = 1 + \frac{Q_5}{L}\mathcal{N}\int_{\vec{d}}\int_0^L \frac{f_\Psi(\vec{d})\,|\vec{F}'(s)|^2 ds}{|\vec{x}-\vec{F}(s)|^2}, \tag{2.25}$$

$$A_i = \frac{Q_5}{L}\mathcal{N}\int_{\vec{d}}\int_0^L \frac{f_\Psi(\vec{d})\,F_i'(s) ds}{|\vec{x}-\vec{F}(s)|^2}, \tag{2.26}$$

where the normalization factor is

$$\mathcal{N}^{-1} = \int_{\vec{d}} f_\Psi(\vec{d}). \tag{2.27}$$

This is a mapping from a quantum system to a set of semiclassical geometries, and we shall see in section 4 that it shouldn't be applied to an arbitrary state, or more generally to an arbitrary density matrix, as this may yield unphysical spacetimes. In section 4 we propose a metric operator in the CFT, the eigenstates of which can be associated to microstate geometries using the prescription above.

3. Asymptotic expansion of a basis state

We wish to determine how the microstate geometry (2.24), (2.25), (2.26) corresponding to a given basis state (2.18) appears to an asymptotic observer. To accomplish this, we shall expand f_5, given by (2.24), as a power series in the inverse radial coordinate $\frac{1}{r}$. For completeness, we also compute the expansion of f_1 in appendix B. For $r \gg |\vec{F}(s)|$ we can expand

$$|\vec{x}-\vec{F}(s)|^{-2} = r^{-2}\left(1 - \frac{2\vec{r}\cdot\vec{F}(s) - |\vec{F}(s)|^2}{r^2}\right)^{-1} = \frac{1}{r^2}\sum_{n=0}^{\infty}\left(\frac{2\vec{r}\cdot\vec{F}(s) - |\vec{F}(s)|^2}{r^2}\right)^n. \tag{3.1}$$

Plugging this into (2.24) and expanding the binomial we get

$$f_5 = 1 + \frac{Q_5}{L}\mathcal{N}\frac{1}{r^2}\sum_{n=0}^{\infty}\frac{1}{r^{2n}}\sum_{p=0}^{n}\binom{n}{p}(-1)^p 2^{n-p} r^{n-p}\int_0^L\int_{\vec{d}} f(\vec{d})(\vec{e}\cdot\vec{F}(s))^{n-p}|\vec{F}(s)|^{2p}, \tag{3.2}$$

where $\vec{r} \equiv r\vec{e}$ and $|\vec{e}|^2 = 1$. To make the powers of $\frac{1}{r}$ more explicit, we define a new summation index $l \equiv n+p$, which runs from 0 to infinity. Eliminating n, we see that p now runs from 0 to $\left[\frac{l}{2}\right]$. To make the integral more explicit, we also eliminate $\vec{F}(s)$ using (2.11) . This gives

$$f_5 = 1 + \frac{Q_5}{r^2}\sum_{l=0}^{\infty}\left(\frac{\mu}{r}\right)^l \sum_{p=0}^{\left[\frac{l}{2}\right]}(-1)^p 2^{\frac{l}{2}-2p}\binom{l-p}{p}\sum_{\substack{k_1,\ldots,k_p\\ l_1,\ldots,l_p\\ m_1,\ldots,m_{l-2p}}}\frac{\delta(\sum_i(k_i+l_i)+\sum_j m_j)}{\sqrt{|\prod_i k_i l_i \prod_j m_j|}}\cdot$$

$$\cdot\,\mathcal{N}\int_{\vec{d}}\prod_{s=1}^{\infty}\prod_{a=1}^{4} e^{-d_s^a d_s^{a*}}\,(d_s^a d_s^{a*})^{N_s^a}\prod_{i=1}^{p}\left(\vec{d}_{k_i}\cdot\vec{d}_{l_i}\right)\prod_{j=1}^{l-2p}\left(\vec{e}\cdot\vec{d}_{m_j}\right), \tag{3.3}$$

where the integral over s gave rise to the Kronecker delta. The integral can only be non-zero when the number of $\vec{d}$'s is even, so we can write $l \equiv 2n$, which gives

$$f_5 = 1 + \frac{Q_5}{r^2} \sum_{n=0}^{\infty} \left(\frac{\mu}{r}\right)^{2n} \sum_{p=0}^{n} (-1)^p 2^{n-2p} \binom{2n-p}{p} \sum_{\substack{k_1,\ldots,k_p \\ l_1,\ldots,l_p \\ m_1,\ldots,m_{2(n-p)}}} \frac{\delta(\sum_i (k_i + l_i) + \sum_j m_j)}{\sqrt{|\prod_i k_i l_i \prod_j m_j|}} \cdot$$

$$\cdot \mathcal{N} \int_{\vec{d}} \prod_{s=1}^{\infty} \prod_{a=1}^{4} e^{-d_s^a d_s^{a*}} \left(d_s^a d_s^{a*}\right)^{N_s^a} \prod_{i=1}^{p} \left(\vec{d}_{k_i} \cdot \vec{d}_{l_i}\right) \prod_{j=1}^{2(n-p)} \left(\vec{e} \cdot \vec{d}_{m_j}\right). \qquad (3.4)$$

In the above all the remaining integrals are gaussian. However, the combinatorics of the indices k_i, l_i and m_j quickly become untractable and we have been unable to find a closed form expression for the n^{th} level of the expansion. In appendix A we present a procedure that can in principle be used compute any given order, though it quickly becomes very tedious for higher orders.

Lacking a general closed form for the expansion, we can at least compute the first few nontrivial orders. For simplicity, we also take the occupation numbers to be independent of direction in the $\mathbb{R}^4$, i.e. $N_k^a = N_k$. As shown in the appendix, we get

$$f_5 = 1 + \frac{Q_5}{r^2} - 12\frac{Q_5 \mu^4}{r^6} M_2 + 40\frac{Q_5 \mu^6}{r^8} M_3 + \mathcal{O}(\frac{1}{r^{10}}), \qquad (3.5)$$

where we have defined the multipoles

$$M_k = \sum_{m=1}^{\infty} \frac{(N_m)^k}{m^k}. \qquad (3.6)$$

As argued in the appendix, the multipole M_k will first appear in the coefficient of $\frac{1}{r^{2k+2}}$ in the expansion. The measurability of these higher order terms depends on how they scale as N is taken to infinity. The average occupation numbers are given by Bose-Einstein statistics, a fact we shall show in section 6; for now we just take this as a given and find

$$\langle M_k \rangle = \sum_{m=1}^{\infty} \frac{1}{m^k} \frac{1}{(e^{\beta m} - 1)^k} \approx \sum_{m=1}^{\infty} \frac{1}{m^{2k}} \frac{1}{\beta^k} \sim N^{\frac{k}{2}}, \qquad (3.7)$$

where the inverse temperature scales as $\beta \propto N^{-\frac{1}{2}}$. We also know that $r \propto N^{\frac{1}{4}}$ and $Q_5 \propto \sqrt{N}$, from which it follows that the combination $\frac{Q_5 M_k}{r^{2k+2}}$ is remains finite in the limit $N \to \infty$, and therefore the higher order terms in the expansion are measurable for an observer that can make measurements with sufficient precision. Since f_5 appears directly in the metric, an asymptotic observer can measure these multipoles and retrieve some data about the CFT state.

To be more precise, an asymptotic observer can measure the multipole M_k by measuring the $(2k+2)^{th}$ derivative of the metric, or a suitable invariant composed of the derivatives. If such a measurement is made with a machine of finite spatial size λ, the resolution of the machine must be at least $\lambda/2k$. Since any measurement is bounded by the Planck scale, this gives a condition

$$\frac{\lambda}{2k} > l_p^{(6)}, \qquad (3.8)$$

where the six-dimensional Planck length is defined in terms of the 6D Newton's constant and the 6D string coupling as $(l_p^{(6)})^4 = G_6 = g_6^2$. If the size of the measurement apparatus is $\lambda = \gamma R_{\mathrm{AdS}_3}$, we get

$$k \lesssim \frac{\gamma R_{\mathrm{AdS}_3}}{\sqrt{g_6}} = \gamma \frac{\sqrt{g_6} N^{\frac{1}{4}}}{\sqrt{g_6}} = \gamma N^{\frac{1}{4}}. \tag{3.9}$$

This gives a limit to how much CFT data an asymptotic observer with sufficient ingenuity can measure. However, this bound is very likely to be too generous; measuring multipoles of order $k \sim N^{\frac{1}{4}}$ involves high energies, the backreaction of which on the geometry cannot be ignored. Thus it is no longer sufficient to work in the $\frac{1}{2}$-BPS sector without taking into account the α' and g_s corrections, which are likely to impose a tighter bound on how many multipoles are measurable. In this note we will not attempt to analyse this in more detail.

4. The metric operator

We shall now explain our earlier statement that the map (2.24), (2.25), (2.26) does not extent to all the states $|\Psi\rangle$ in the Hilbert space. Consider a superposition of two very different states, say

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle, \tag{4.1}$$

$$\text{with} \quad |\psi_1\rangle = \prod_{a=1}^{4} \frac{1}{\sqrt{(N/4)!}} \left(c_1^{a\dagger}\right)^{\frac{N}{4}} |0\rangle, \quad \text{and} \quad |\psi_2\rangle = \prod_{a=1}^{4} c_{N/4}^{a\dagger}|0\rangle.$$

Note that neither of these states is typical in any sense, but they serve to illustrate the issue; we will deal with the full thermal ensemble of states in section 6. We immediately find the multipoles M_k in these states as[2]

$$M_k^{\psi_1} = \left(\frac{N}{4}\right)^k, \quad M_k^{\psi_2} = \frac{1}{\left(\frac{N}{4}\right)^k}. \tag{4.2}$$

Since (2.19) and (2.24) are linear[3] in the density matrix, the multipoles of the state $|\Psi\rangle$ are given by $M_k^{\Psi} = \frac{1}{2}(M_k^{\psi_1} + M_k^{\psi_2})$, which is very different from both $M_k^{\psi_1}$ and $M_k^{\psi_2}$. This is not problematic from the CFT point of view, but the spacetime interpretation presents a problem. As soon as an observer measures any of the multipoles in the spacetime, standard measurement theory arguments state that the universe is projected into either of the two states ψ_1 or ψ_2. But the three geometries differ from each other at scales which are easily measurable and therefore 'jumping' between these metrics based on one measurement is not physically acceptable. Because of this problem we need to develop a criterion that establishes when a state can be mapped into a microstate geometry using (2.24), (2.25), (2.26), and when it's not reasonable to associate a semiclassical metric to a state in the CFT.

[2] Due to the non-typicality, these don't scale as $N^{\frac{k}{2}}$ like they would in a typical state. Indeed, ψ_1 has the maximal possible multipoles, while ψ_2 has the smallest possible multipoles.

[3] The density matrix for Ψ will have cross terms $|\psi_1\rangle\langle\psi_2|$ and $|\psi_2\rangle\langle\psi_1|$. However, we shall show in section 5 that these will have minimal contribution to the phase space distribution and will not affect the multipoles. Therefore the distribution is the sum of the distributions of ψ_1 and ψ_2.

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4.1 The metric operator and eigenstates

We shall now define the general multipole operator[4] as

$$\hat{M}(k) \equiv \sum_{m=1}^{\infty} \frac{1}{m^k} \hat{N}_m^k = \sum_{m=1}^{\infty} \frac{1}{m^k} \left(c_m^\dagger c_m \right)^k , \tag{4.3}$$

which is simply the quantum version of (3.6). Note that we are suppressing the $\mathbb{R}^4$ indices.

Next we need to define what we mean by approximate eigenstates of the operator $\hat{M}(k)$. From the definition it is clear that the only exact eigenstates are the basis states (2.18), while any superposition is necessarily not an eigenstate. This is too restricting; rather we wish to introduce a coarse graining to correspond to the limited measurement precision of an observer. To do this, for an arbitrary state $|\Psi\rangle$ we introduce the functional

$$E[M(k)] = \mathrm{Tr} \left[\hat{\rho}_\Psi \left(\hat{M}(k) - M(k) \right)^2 \right] , \tag{4.4}$$

and we shall call the function that minimizes this functional $M_\Psi(k)$. Thus armed, we say that $|\Psi\rangle$ is an eigenstate of $\hat{M}(k)$ with eigenvalue function $M_\Psi(k)$ and accuracies $\{\epsilon_k\}$, iff

$$\frac{\sqrt{E[M_\Psi(k)]}}{M_\Psi(k)} < \epsilon_k, \quad \text{for all } k. \tag{4.5}$$

Note that if $|\Psi\rangle = |\{N_k\}\rangle$ is a basis state, then $E[M_\Psi(k)] = 0$, with $M_\Psi(k)$ given by (3.6), and (4.5) is trivially satisfied.

With this definition, we are finally in a position to state our proposal in a definite form:

The states in the CFT that have good dual description in terms of a unique metric are the ones that are approximate eigenstates of the operator $\hat{M}(k)$.

In this sense we can also call $\hat{M}(k)$ a 'metric' operator: its eigenstates are the only ones that can be mapped to semi-classical spacetimes with unique metrics, and its eigenvalue functions specify the multipoles present in the asymptotic expansion of the metric and allow an observer to reconstruct the metric up to some measurement precision.

4.2 Explicit example

Before closing this section, we wish to illustrate this formalism by considering an explicit example. We choose the state to be a superposition of two basis states: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\{N_{m1}\}\rangle + |\{N_{m2}\}\rangle)$. The expectation values in (4.4) are easily evaluated and yield

$$\begin{aligned} \langle\Psi|\hat{M}(k)|\Psi\rangle &= \frac{1}{2}\left(\sum_{m=1}^{\infty} \frac{N_{m1}^k}{m^k} + \sum_{m=1}^{\infty} \frac{N_{m2}^k}{m^k} \right), \\ \langle\Psi|\hat{M}(k)^2|\Psi\rangle &= \sum_{m,n=1}^{\infty} \frac{1}{m^k n^k} \langle\Psi|\hat{N}_m^k \hat{N}_n^k|\Psi\rangle = \frac{1}{2}\left(\left(\sum_{m=1}^{\infty} \frac{N_{m1}^k}{m^k} \right)^2 + \left(\sum_{m=1}^{\infty} \frac{N_{m2}^k}{m^k} \right)^2 \right). \end{aligned} \tag{4.6}$$

[4]The idea of using a formalism like this to determine which states can be mapped to unique semiclassical geometries was first used in the setting of $\frac{1}{2}$-BPS sector of $\mathcal{N} = 4$ SU(N) Yang-Mills in [4].

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Plugging these into the functional (4.5), we can write it as

$$E[M(k)] = \left(M(k) - \frac{1}{2} \sum_{m=1}^{\infty} \frac{N_{m1}^k + N_{m2}^k}{m^k} \right)^2 + \frac{1}{4} \left(\sum_{m=1}^{\infty} \frac{N_{m1}^k}{m^k} - \sum_{m=1}^{\infty} \frac{N_{m2}^k}{m^k} \right)^2 . \tag{4.7}$$

This is minimized by choosing $M_\Psi(k) = \frac{1}{2} \sum \frac{N_{m1}^k + N_{m2}^k}{m^k} = \frac{1}{2}(M_{k,\{N_{m1}\}} + M_{k,\{N_{m2}\}})$, i.e. average of the multipoles of the two states. However, the functional never vanishes and the condition (4.5) can be written as

$$\frac{|M_{k,\{N_{m1}\}} - M_{k,\{N_{m1}\}}|}{M_{k,\{N_{m1}\}} + M_{k,\{N_{m2}\}}} < \epsilon_k , \tag{4.8}$$

which gives a condition for how much the multipoles of the two states can differ if $|\Psi\rangle$ is to be an eigenstate with accuracy ϵ_k. For the superposition considered at the beginning of this section, (4.1), the ratio above is of order one, and therefore this state is far from being an eigenstate.

5. More asymptotic expansions

We now wish to find the asymptotic expansion for a general state in the theory, rather than just for basis states. Of course, for any state we need to check that it is an approximate eigenstate of $\hat{M}(k)$ before we can trust this expansion. A general superposition is given by

$$|\psi\rangle = \sum_w \alpha_w \prod_{k=1}^{\infty} \prod_{a=1}^{4} \frac{\left(c_k^{a\dagger}\right)^{N_k^{a,w}}}{\sqrt{N_k^{a,w}!}} |0\rangle, \text{with} \quad \sum_{k=1}^{\infty} \sum_{a=1}^{4} k N_k^{a,w} = N \ \forall w, \quad \text{and} \ \sum_w |\alpha_w|^2 = 1, \tag{5.1}$$

where w indexes the states in the superposition. The phase space distribution can again be computed, and yields

$$\begin{aligned} f(\vec{d}) &= \sum_{w,w'} \alpha_w \alpha_{w'}^* \prod_{k=1}^{\infty} \prod_{a=1}^{4} e^{-d_k^a d_k^{a*}} \, (d_k^a)^{N_k^{a,w}} \, (d_k^{a*})^{N_k^{a,w'}} \\ &= \sum_{w,w'} \alpha_w \alpha_{w'}^* \prod_{k=1}^{\infty} \prod_{a=1}^{4} e^{-(\rho_k^a)^2} (\rho_k^a)^{(N_k^{a,w} + N_k^{a,w'})} e^{i\phi_k^a (N_k^{a,w} - N_k^{a,w'})}, \end{aligned} \tag{5.2}$$

where in the second equality we have switched to polar coordinates. Thus we can see that all angular dependence in the phase space distribution is due to interference terms between different basis states. Following the recipe laid out in section 3, we can expand f_5 in $\frac{1}{r}$ to get

$$\begin{aligned} f_5 = 1 + \frac{Q_5}{r^2} \sum_{w,w'} \alpha_w \alpha_{w'}^* \sum_{l=0}^{\infty} \left(\frac{\mu}{r}\right)^l \sum_{p=0}^{[\frac{l}{2}]} (-1)^p 2^{\frac{l}{2} - 2p} \binom{l-p}{p} \sum_{\substack{k_1,\ldots,k_p \\ l_1,\ldots,l_p \\ m_1,\ldots,m_{l-2p}}} \frac{\delta(\sum_i (k_i + l_i) + \sum_j m_j)}{\sqrt{|\prod_i k_i l_i \prod_j m_j|}} \cdot \\ \cdot \mathcal{N} \int_{\vec{d}} \prod_{k=1}^{\infty} \prod_{a=1}^{4} e^{-d_k^a d_k^{a*}} \, (d_k^a)^{N_k^{a,w}} \, (d_k^{a*})^{N_k^{a,w'}} \prod_{i=1}^{p} \left(\vec{d}_{k_i} \cdot \vec{d}_{l_i} \right) \prod_{j=1}^{l-2p} \left(\vec{e} \cdot \vec{d}_{m_j} \right) . \end{aligned} \tag{5.3}$$

Though analyzing this in detail is untractable, we can still make some interesting observations. Since all the terms in the phase space distribution (5.2) do not in general have an even number of d's, we see that the summation index l does not need to be even anymore, and thus the expansion now has terms that are odd in $\frac{1}{r}$. The origin of these terms is completely due to interference between basis states.

5.1 Expansion for coherent states

Analyzing the measurability of the odd terms in (5.3) is difficult when working in the basis of eigenstates of excitation numbers. However, using coherent states we can explicitly show that these terms can be measurable. The phase space distribution corresponding to a coherent state was written down in (2.22), and using it we can once again expand (2.24) to get

$$f_5 = 1 + \frac{Q_5}{r^2} + 4\frac{Q_5\mu^2}{r^4}\sum_{m=1}^{\infty}\frac{1}{m}\left[(\vec{\tilde{d}}_m\cdot\vec{e})(\vec{\tilde{d}}_{-m}\cdot\vec{e}) - (\vec{\tilde{d}}_m\cdot\vec{\tilde{d}}_{-m})\right] + \tag{5.4}$$
$$+\sqrt{2}\frac{Q_5\mu^3}{r^5}\sum_{k,l,m}\frac{\delta(k+l+m)}{\sqrt{|klm|}}\left[2(\vec{\tilde{d}}_m\cdot\vec{e})(\vec{\tilde{d}}_k\cdot\vec{e})(\vec{\tilde{d}}_l\cdot\vec{e}) - (\vec{\tilde{d}}_k\cdot\vec{\tilde{d}}_l)(\vec{e}\cdot\vec{\tilde{d}}_m)\right] + \mathcal{O}\left(\frac{1}{r^6}\right).$$

To complete the analysis, we need to show that these odd terms are measurable and that coherent states are approximate eigenstates of $\hat{M}(k)$ and therefore it is sensible to associate semiclassical geometries to them.

Measurability: we need to determine how the $\vec{\tilde{d}}$'s scale as a function of N. To do this, we compute the overlap between the coherent state and an arbitrary basis state. This can be done using (2.18) and (2.22), and gives

$$\langle\{N_k\}|\vec{\tilde{d}}\rangle = \prod_{k=1}^{\infty} e^{-\frac{|d_k|^2}{2}}\frac{(d_k^*)^{N_k}}{\sqrt{N_k!}}. \tag{5.5}$$

To determine which basis state has the largest overlap with the coherent state, we maximize the norm of (5.5) and find

$$|\vec{\tilde{d}}_k| = \sqrt{N_k}. \tag{5.6}$$

We want the $N\to\infty$ limit to be one that leaves inner products like (5.5) unchanged; hence (5.6) tells us the scaling of $\vec{\tilde{d}}_k$. For states near the typical state, $N_k \propto \sqrt{N}$ for small k, and therefore we see that the terms in the expansion remain fixed as N is scaled[5]. This is enough to show that the effects of interference remain observable, even in the $N\to\infty$ limit.

Eigenstates: finally, we need to show that the coherent states are approximate eigenstates of $\hat{M}(k)$. The expectation values of $\hat{M}(k)$ and $\hat{M}(k)^2$ are

$$\mathrm{Tr}(\hat{\rho}\hat{M}(k)) = \sum_{m=1}^{\infty}\frac{1}{m^k}\sum_{\{N_p\}}|\langle\{N_p\}|\vec{\tilde{d}}\rangle|^2\langle\{N_p\}|N_m^k|\{N_p\}\rangle$$

[5]Remember that r scales as $N^{1/4}$ and Q_5 as $\sqrt{N}$

$$= \sum_{m=1}^{\infty} \frac{1}{m^k} e^{-|d_m|^2} \sum_{N_m=0}^{\infty} \frac{|d_m|^{2N_m}}{N_m!} (N_m)^k, \tag{5.7}$$

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}\hat{M}(k)^2) &= \sum_{m,n=1}^{\infty} \frac{1}{m^k n^k} \sum_{\{N_p\}} |\langle \{N_p\}|\vec{\tilde{d}}\rangle|^2 \langle \{N_p\}|N_m^k N_n^k|\{N_p\}\rangle \\
&= \left(\sum_{m=1}^{\infty} \frac{e^{-|d_m|^2}}{m^k} \sum_{N_m=0}^{\infty} \frac{|d_m|^{2N_m}}{N_m!} N_m^k \right)^2 + \\
&+ \sum_{m=1}^{\infty} \frac{e^{-|d_m|^2}}{m^{2k}} \sum_{N_m=0}^{\infty} \frac{|d_m|^{2N_m}}{N_m!} N_m^{2k} - \sum_{m=1}^{\infty} \frac{e^{-2|d_m|^2}}{m^{2k}} \left(\sum_{N_m=0}^{\infty} \frac{|d_m|^{2N_m}}{N_m!} N_m^k \right)^2
\end{aligned} \tag{5.8}$$

Using these, one shows that the functional $E[M(k)]$ in (4.4) can be written as

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}(\hat{M}(k) - M(k))^2) &= \left(M(k) - \sum_{m=1}^{\infty} \frac{1}{m^k} e^{-|d_m|^2} \sum_{n=0}^{\infty} \frac{|d_m|^{2n}}{n!} n^k \right)^2 + \\
&+ \sum_{m=1}^{\infty} \frac{e^{-|d_m|^2}}{m^{2k}} \left[\sum_{n=0}^{\infty} \frac{|d_m|^{2n}}{n!} n^{2k} - e^{-|d_m|^2} \left(\sum_{n=0}^{\infty} \frac{|d_m|^{2n}}{n!} n^k \right)^2 \right]
\end{aligned} \tag{5.9}$$

The function $M_{\tilde{d}}(k)$ is again chosen such that the first square vanishes. Thus it remains to show that the remaining terms yield a negligible contribution. To do this, we note that the sums appearing in the the expression above can be computed as

$$e^{-r} \sum_{n=0}^{\infty} \frac{r^n}{n!} n^k = e^{-r} (r\partial_r)^k \sum_{n=0}^{\infty} \frac{r^n}{n!} = e^{-r} (r\partial_r)^k e^r = \text{Polynomial of order } k \text{ in } r. \tag{5.10}$$

Using this and writing $|d_m|^2 = r_m$, we can write the ratio (4.5) as

$$\frac{\sqrt{E[M_{\tilde{d}}(k)]}}{M_{\tilde{d}}(k)} = \frac{\sqrt{\sum_{m=1}^{\infty} \frac{e^{-r_m}}{m^{2k}} \left[\sum_{n=0}^{\infty} \frac{r_m^n}{n!} n^{2k} - e^{-r_m} \left(\sum_{n=0}^{\infty} \frac{r_m^n}{n!} n^k \right)^2 \right]}}{\sum_{m=1}^{\infty} \frac{1}{m^k} e^{-r_m} \sum_{n=0}^{\infty} \frac{r_m^n}{n!} n^k}. \tag{5.11}$$

The denominator is a polynomial of order k in r_m, while in the numerator the highest order in r_m cancels and one is left with a square root of a polynomial of order $2k-1$ in r_m. Using the scaling (5.6) we then see

$$\frac{\sqrt{E[M_{\tilde{d}}(k)]}}{M_{\tilde{d}}(k)} \sim \frac{\sqrt{|d_m|^{4k-2}}}{|d_m|^{2k}} \sim \frac{1}{|d_m|} \sim \frac{1}{N^{\frac{1}{4}}}, \tag{5.12}$$

showing that for large N this is suppressed and the coherent state is an approximate eigenstate to a high precision.

6. The canonical ensemble

Explicit computations in the microcanonical ensemble involving only states of a fixed total twist N can be complicated. One often used method of circumventing this is to work in a canonical ensemble, fixing the total twist to equal N using a Lagrange multiplier. However, we shall show that this ensemble is not well suited for use with the mapping (2.24), (2.25), (2.26), and this is possibly the reason why in [13] a non-standard entropy was found for the $M = 0$ BTZ black hole.

In the canonical ensemble the thermal density matrix can be written as

$$\hat{\rho} = \sum_{\{N_k\}} \frac{e^{-\beta\hat{N}}|\{N_k\}\rangle\langle\{N_k\}|}{\mathrm{Tr}(e^{-\beta\hat{N}})} = \prod_{k=1}^{\infty}(1 - e^{-\beta k}) \sum_{N_k=0}^{\infty} e^{-\beta k N_k}|k, N_k\rangle\langle k, N_k|, \tag{6.1}$$

where $|k, N_k\rangle = \frac{1}{\sqrt{N_k!}}(c_k^\dagger)^{N_k}|0\rangle$, and β has to be fixed by the condition $\langle\hat{N}\rangle = N$. Note that we're treating all directions as isotropic, and thus suppressing the $\mathbb{R}^4$ index a. The expected occupation numbers and total twist were computed in [13] to give

$$\langle\hat{N}_m\rangle = \mathrm{Tr}(\hat{\rho}\hat{N}_m) = (1 - e^{-\beta m}) \sum_{N_m=0}^{\infty} N_m e^{-\beta m N_m} = \frac{1}{e^{\beta m} - 1}, \tag{6.2}$$

$$\langle\hat{N}\rangle = \sum_{m=1}^{\infty} m\langle\hat{N}_m\rangle = \frac{2\pi^2}{3\beta^2}. \tag{6.3}$$

The second equation fixes the inverse temperature

$$\beta = \pi\sqrt{\frac{2}{3N}}. \tag{6.4}$$

In addition to these we will need the expectation values of higher powers of the occupation numbers. For $\beta m \ll 1$, we can find them by approximating the sum by an integral, which yields

$$\langle\hat{N}_m^k\rangle = (1 - e^{-\beta k}) \sum_{N_m=0}^{\infty} N_m^k e^{-\beta m N_m} \approx (1 - e^{-\beta k}) \int_0^\infty dN_m N_m^k e^{-\beta m N_m} \approx \frac{k!}{\beta^k m^k}. \tag{6.5}$$

6.1 Limitations of the canonical ensemble

There is a problem with using the canonical ensemble with the CFT-to-gravity mapping (2.24), (2.25), (2.26), as can be seen by computing the standard deviation to mean ratio of the occupation numbers[6]:

$$\frac{\sigma(\hat{N}_k)}{\langle\hat{N}_k\rangle} = \frac{\sqrt{\langle\hat{N}_k^2\rangle - \langle\hat{N}_k\rangle^2}}{\langle\hat{N}_k\rangle} = e^{\frac{\beta k}{2}}. \tag{6.6}$$

This doesn't vanish in the $N \to \infty$ limit, and is an indication that the fluctuations in the occupation numbers are always large. This doesn't invalidate the ensemble as such, since one can show that the fluctuations in the total twist $\langle\hat{N}\rangle$ are of the order $N^{-\frac{1}{4}}$ and therefore the ensemble samples only states of twist N to a good degree. However, in using

[6]This looks different from what (6.5) would give, as this is an exact result. To leading order (6.5) will give the same result.

the CFT-to-gravity mapping the fluctuations in $\hat{N}_k$'s are of paramount importance, as they lead to large fluctuations in the multipoles $\hat{M}_k$, which in turn lead to superpositions of states of very different metrics, as in the example at the beginning of section 4. Thus, this thermal state should not be mapped to a geometry at all. Indeed, we can check that this density matrix does not satisfy (4.5) and therefore does not pass our criterion. We can use (6.5) to compute the expectation value of $\hat{M}(k)$;

$$\mathrm{Tr}(\hat{\rho}\hat{M}(k)) = \sum_{m=1}^{\infty} \frac{1}{m^k} \mathrm{Tr}(\hat{\rho}\hat{N}_m^k) = \sum_{m=1}^{\infty} \frac{\langle N_m^k \rangle}{m^k} \approx \sum_{m=1}^{\infty} \frac{k!}{\beta^k m^{2k}} = \frac{k!\zeta(2k)}{\beta^k}, \tag{6.7}$$

and the expectation value of the square

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}\hat{M}(k)^2) &= \sum_{m,n=1}^{\infty} \frac{1}{m^k n^k} \langle \hat{N}_m^k \hat{N}_n^k \rangle \approx \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{k!^2}{\beta^{2k} m^{2k} n^{2k}} + \sum_{m=1}^{\infty} \frac{(2k)!}{\beta^{2k} m^{4k}} \\ &= \left(\sum_{m=1}^{\infty} \frac{k!}{\beta^k m^{2k}} \right)^2 + \sum_{m=1}^{\infty} \frac{(2k)! - k!^2}{\beta^{2k} m^{4k}} = \left(\frac{k!\zeta(2k)}{\beta^k} \right)^2 + \frac{(2k)! - k!^2}{\beta^{2k}} \zeta(4k). \end{aligned} \tag{6.8}$$

Putting these two results together we can again compute functional (4.4):

$$E[M(k)] = \left(M(k) - \frac{k!\zeta(2k)}{\beta^k} \right)^2 + \frac{(2k)! - k!^2}{\beta^{2k}} \zeta(4k), \tag{6.9}$$

which is minimized by choosing $M_{\hat{\rho}}(k) = \frac{k!\zeta(2k)}{\beta^k}$. However, the second term will not vanish, and moreover is not small by any criterion as can be seen by computing the ratio in (4.5):

$$\frac{\sqrt{E[M_{\hat{\rho}}(k)]}}{M_{\hat{\rho}}(k)} \approx \sqrt{\frac{(2k)!}{k!^2} - 1} > 1, \tag{6.10}$$

which is greater than any reasonable measurement precision ϵ_k. Thus the mixed thermal state is not an approximate eigenstate of $\hat{M}(k)$ and should not be associated to any semi-classical geometry.

6.2 A restricted canonical ensemble?

Due to the limitations stated above, we would like to in some way restrict the canonical ensemble in order to curb down the fluctuations in the multipoles. The most obvious way of doing this would be to fix the first p excitation numbers $N_1, \ldots, N_p$ to be given by the Bose-Einstein excitation numbers (6.11), either by hand or using Lagrange multipliers. This would be in close analogy with what was found in the LLM case in [6], where one had to restrict the ensemble by fixing the highest excitation in the system to curb the fluctuations in the ensemble. We shall explore this and other similarities with the LLM case in the discussion section. Unfortunately, in our case this method fails to sufficiently stabilize the ensemble, though we feel it is still interesting to present the analysis and investigate why this is so.

Thus we begin by fixing

$$N_m \equiv N_c^{(m)} = \frac{1}{e^{\beta m} - 1}, \tag{6.11}$$

so that the the density matrix reduces to

$$\hat{\rho} = |1, N_c^{(1)}\rangle\langle 1, N_c^{(1)}|\otimes\ldots\otimes|p, N_c^{(p)}\rangle\langle p, N_c^{(p)}|\otimes\left(\prod_{k=p+1}^{\infty}(1-e^{-\beta k})\sum_{N_k=0}^{\infty}e^{-\beta k N_k}|k, N_k\rangle\langle k, N_k|\right). \tag{6.12}$$

Using this density matrix it is clear that the first p excitation numbers do not fluctuate at all, and the fluctuations of the higher N_m's are as in the unrestricted ensemble. Using the results from the previous subsection it is easy to compute the functional (4.4), which gives

$$E[M(k)] = \left[M(k) - \left(\sum_{m=1}^{p}\frac{(N_c^{(m)})^k}{m^k} + \sum_{m=p+1}^{\infty}\frac{k!}{\beta^k m^{2k}}\right)\right]^2 + \sum_{m=p+1}^{\infty}\frac{(2k)! - k!^2}{\beta^{2k}m^{4k}}. \tag{6.13}$$

Choosing $M(k)$ to minimize the first square, we can compute the ratio

$$\frac{\sqrt{E[M_{\hat{\rho}}(k)]}}{M_{\hat{\rho}}(k)} \approx \frac{\sqrt{[(2k)! - k!^2]\zeta_{p+1}(4k)}}{\zeta(2k) + (k! - 1)\zeta_{p+1}(2k)}, \tag{6.14}$$

where we defined the partial zeta function as $\zeta_{p+1}(k) = \sum_{m=p+1}^{\infty} m^{-k}$. We may estimate $\zeta_{p+1}(k)$ from below by $\int_{p+1}^{\infty}\frac{dm}{m^k}$ and from above by $\int_{p}^{\infty}\frac{dm}{m^k}$, from which we find

$$\frac{1}{k-1}\frac{1}{(p+1)^{k-1}} < \zeta_{p+1}(k) < \frac{1}{k-1}\frac{1}{p^{k-1}}. \tag{6.15}$$

For small values of k (6.14) does not depend on N, and the fluctuations are small with a suitably chosen p. To see this, we estimate

$$\left(\frac{\sqrt{E[M_{\hat{\rho}}(k)]}}{M_{\hat{\rho}}(k)}\right)^2 < [(2k)! - k!^2]\zeta_{p+1}(4k) \lesssim \frac{(2k)! - k!^2}{(4k-1)p^{4k-1}} < \epsilon_k^2, \tag{6.16}$$

which can always be made smaller than the given measurement precision ϵ_k with a suitably chosen p, without p having to scale with N.

The trouble arises for large values of k, i.e. $k \propto N^{\alpha}$, as an observer can optimally measure multipoles up to $k \sim N^{1/4}$. Using (6.15) it can be shown that for the fluctuations to be small, one needs to choose $p \gg k$; a value so high that almost all the states are projected out of the ensemble, invalidating the statistical treatment of the system.

We have been unable to find a better method of stabilising the multipoles in the canonical ensemble, as the fluctuations in the excitation numbers are quite severe. However, one possible resolution to this problem might be that, although naively an observer is able to measure multipoles up to order $k \sim N^{1/4}$, this might not hold after a more thorough analysis. The reason for this is that when an observer measures high multipoles, high energies are needed and the backreaction of these should not be neglected. Also, for low energies it is safe to work within the $\frac{1}{2}$-BPS sector, but for large energies one expects g_s and α' corrections, which might induce a much stricter limit than $k \sim N^{1/4}$ for the measurable multipoles. If this was the case, the method of restricting the fluctuations described here could be enough to stabilise the ensemble sufficiently; a possibility we shall not analyse in more detail in this note.

7. Discussion

In this note we proposed a criterion that a Ramond ground state in the D1-D5 CFT has to satisfy in order to have a semi-classical gravity dual. This proposal was based on the observation that the data characterizing the CFT state manifests itself as a set of multipoles in the gravity side. Thus any CFT state having a semi-classical gravity dual has to be such that the multipoles associated to it do not have a large quantum variance. In particular, we showed that the density matrix associated to the canonical ensemble is not 'sufficiently classical' to admit a semi-classical description, and analysed a possible way of modifying the ensemble to curb these fluctuations.

Furthermore we showed that while our criterion restricts the states that can have semi-classical duals, certain purely quantum mechanical aspects can be manifest in the semi-classical gravity dual. An example of this is the observation that quantum interference in the CFT can give rise to new, measurable, terms in the asymptotic expansion of the metric.

Comparison with LLM: since the story proposed in this note closely parallels the one developed in [3, 4, 6] for the LLM system, it is interesting to analyse the similarities and differences in these systems.

In both cases the relevant states in the CFT's can be described in terms of excitations in a harmonic potential; only in the LLM case the excitations are fermionic. Thus a basis state is specified by an ordered set of excitation numbers: $\lambda_1 < \ldots < \lambda_N$, and in [3] we showed that in the expansion of the metric these integers appear in moments $M_k^{LLM} = \lambda_1^k + \ldots + \lambda_N^k$, which should be compared with the multipoles (3.6) found here. Thus in the LLM case it is the highest excitations that contribute the most, while in our case the lowest twists are most strongly manifest in the gravity side. This difference is presumably due to the fractionalization present in the D1-D5 system. In both cases, however, the CFT data is arranged into a set of moments/multipoles in the gravity side. This analogy extends to superpositions; in both cases interference terms can be measurable for an asymptotic observer, and some terms in the metric expansion only appear for states that are superpositions of occupation number eigenstates.

Another similarity between the two systems is apparent in the treatment of the canonical ensemble. In the LLM case it was found that states with a few highly excited particles, though few in number, contributed disproportionably to the ensemble. Therefore the ensemble was modified by fixing the highest excitation to be a given number N_c [6], and the fluctuations in this modified ensemble were sufficiently constrained to yield the correct stretched horizon for the superstar geometry of [28]. In our case, we found that the fluctuations in the first excitation numbers rendered the ensemble ill-suited for use with the CFT-to-gravity mapping, and tried to solve this by fixing the first excitation numbers[7]. Unfortunately, we found that to stabilise the high multipoles, one has to fix so many excitations that one loses the statistical description of the system.

[7]The fact that in the LLM case it was sufficient to fix only one excitation can be traced back to the fact that the excitations are ordered, and thus fixing the highest will also affect the others.

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One final difference between the two systems is that, owing to the fact that in LLM one has a two dimensional phase space and fermionic excitations, in LLM one can compute the entropy of any spacetime geometry in a very elegant manner. It is not clear if this can be done in our case, though it would be very interesting if it could be done.

Acknowledgments

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A. Combinatorics

In this appendix we'll provide a prescription for computing an arbitrary order of the integral in (3.4). Thus we need to compute

$$\sum_{p=0}^{n}(-1)^p 2^{n-2p}\binom{2n-p}{p}\sum_{\substack{k_1,\ldots,k_p\\ l_1,\ldots,l_p\\ m_1,\ldots,m_{2(n-p)}}}\frac{\delta(\sum_i(k_i+l_i)+\sum_j m_j)}{\sqrt{|\prod_i k_i l_i \prod_j m_j|}}\cdot\mathcal{I}_{\{N_s^a\}}\times \tag{A.1}$$
$$\times\left[\prod_{i=1}^{p}\left(\vec{d}_{k_i}\cdot\vec{d}_{l_i}\right)\prod_{j=1}^{2(n-p)}\left(\vec{e}\cdot\vec{d}_{m_j}\right)\right],$$

where we have defined the functional integral

$$\mathcal{I}_{\{N_s^a\}}[g(\vec{d})]\equiv\mathcal{N}\int_{\vec{d}}\prod_{s=1}^{\infty}\prod_{a=1}^{4}e^{-d_s^a d_s^{a*}}\left(d_s^a d_s^{a*}\right)^{N_s^a}g(\vec{d}). \tag{A.2}$$

A basic property of $\mathcal{I}_{\{N_s^a\}}$ is that it factorizes in s and a, and we can compute[8]

$$\mathcal{I}\left[(d_k^a d_{-k}^a)^r\right]=\frac{1}{\pi N_k^a!}\int_{d_k^a,d_k^{a*}}e^{-d_k^a d_k^{a*}}\left(d_k^a d_k^{a*}\right)^{N_k^a+r}=N_k^a(N_k^a+1)\ldots(N_k^a+r-1)=(N_k^a)_r, \tag{A.3}$$

where $(x)_n$ is the Pochhammer symbol.

General method: we see that the integral we need to compute is simply a product of gaussian integrals, made complicated by the combinatorics of the indices. The integral clearly can be non-zero only when for every index q there is corresponding index $-q$, i.e. the $2n$ indices $\{k_i, l_i, m_j\}$ are split into pairs and there are thus only n free indices. Let us first treat the case where no two pairs share the same value $|q|$. Thus the set of indices is $\{q_1,\ldots,q_N,-q_1,\ldots,-q_N\}$. The number of times each of these terms appears in the sums

[8] Actually, the computation gives $(N_k^a+1)_r$, but to properly account for the anti-normal ordering prescription we need to translate $d_k^a d_k^{a*}\to d_k^a d_k^{a*}-1$, after which we get $(N_k^a)_r$. See [13] for more details.

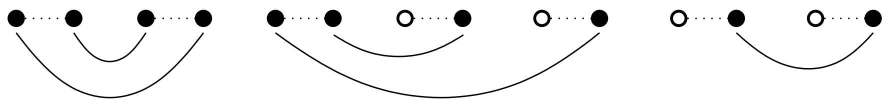

Figure 1: Graphical method for writing the argument of the functional integral. Portrayed is the $n = 5$, $p = 3$ case and one possible pairing.)

over $\{k_i, l_i, m_j\}$ is $2n!$, since k_1 can be any of the $\pm q_i$, k_2 has $2n-1$ options and so on. However, this would completely fix the ordering of the indices, which we do not want to do; we divide by $n!$, so that $q_1, \dots, q_N$ are unordered. Thus, we should always have a total of $\frac{(2n)!}{n!}$ terms with all the q_i different.

Next we need to address how the pairings are distributed among the indices $\{k_i, l_i, m_j\}$. All distributions are clearly not equal, as can be seen from the argument of the functional integral in (A.1). The clearest way of keeping track of all possibilities is a graphical representation, and in figure 1 we have illustrate the $n = 5$, $p = 3$ case. In the figure each solid circle corresponds to a d and each empty circle corresponds to an e. The dots between two circles indicate inner product, i.e. contraction of the $\mathbb{R}^4$ indices.

We need to sum over all possible pairings of indices; we've have drawn one such pairing into the figure, showing with the looping lines which indices form pairs. We also need to keep track of the $\mathbb{R}^4$ index structure; by following the lines and the inner products in figure 1, we see that the 'strings' created by these lines come in two varieties: 'closed' and 'open'. By closed we mean any loop such as the one connecting the left four d's in the figure, while open loops always end in e's (empty circles) on both ends. Thus there is one closed and two open loops in the figure.

Next we need to see how these loops contribute; this is easiest to do by considering the example in the figure and computing the contribution from the closed loop and the middle (open) loop. Due to the factorization these can be computed separately, and we get

$$\begin{cases} \text{Closed (left):} & \mathcal{I}[(\vec{d}_{k_1} \cdot \vec{d}_{l_1})(\vec{d}_{-l_1} \cdot \vec{d}_{-k_1})] = \sum_{a,b=1}^4 N^a_{k_1} N^b_{l_1} \delta_{ab} = 4 N_{k_1} N_{l_1}, \\ \text{Open (middle):} & \mathcal{I}[(\vec{d}_{k_3} \cdot \vec{d}_{l_3})(\vec{e} \cdot \vec{d}_{-l_3})(\vec{e} \cdot \vec{d}_{-k_3})] = \sum_{a=1}^4 N^a_{k_3} N^a_{l_3} e_a^2 = N_{k_3} N_{l_3}, \end{cases} \tag{A.4}$$

from which we see that closed loops get a factor of 4 from the index structure, while open ones get $\vec{e}^{\,2}$, which is unity. (Note that we are always dealing with the case where the occupation numbers don't depend on direction, i.e. $N^a_k = N_k$.) Now we are ready to deal with all the cases where no two pairs coincide.

The case where two or more pairs coincide is very similar; the only real difference is the the number of terms we expect. Let us assume we have n pairs, two of which coincide, i.e. $q_i = q_j$ for some i and j. In this case the total number of terms is $\frac{(2n)!}{(n-2)!2!2!}$, where $\frac{(2n)!}{2!2!}$ is the total number of terms[9]. We again divide by $(n-2)!$ to make sure the q_k are unordered. More complicated cases can also be worked out similarly.

[9] For example, when no pairs coincided, the term $d_{q_i} d_{q_j}$ could come from $k_1 = q_i$ and $k_2 = q_j$, or $k_1 = q_j$ and $k_2 = q_i$. Now that $q_i = q_j$ there is only one term, $k_1 = k_2 = q_i = q_j$; thus we need to divide by the degeneracies.

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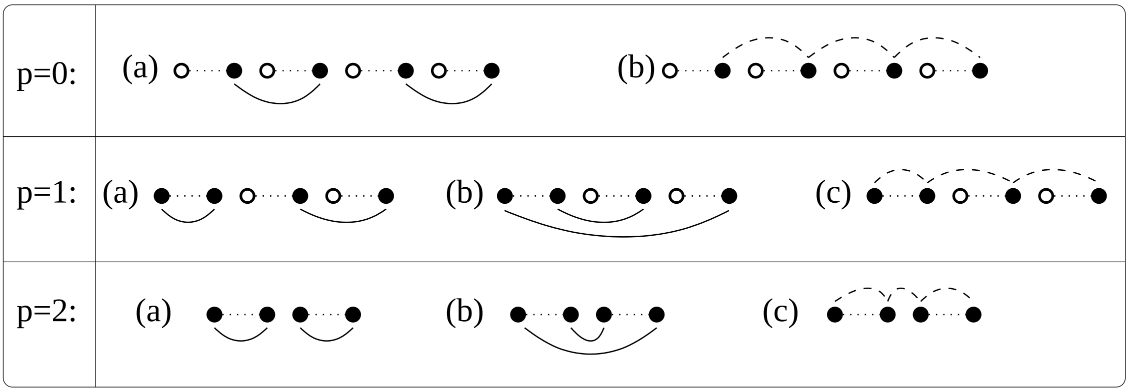

Figure 2: All possible 'topologically' distinct pairings for $n = 2$. The dashes lines above the dots indicate that the dots connected by these lines share the same index (up to sign).

A.1 The $n = 2$ case explicitly

To illustrate the method explained above, we shall now work out the case $n = 2$ in some detail[10]. We see from (A.1) that we need to compute

$$4 \sum_{m_1,\ldots,m_4} \frac{\delta(\ldots)}{\sqrt{|m_1 m_2 m_3 m_4|}} \mathcal{I}[(\vec{e}\cdot\vec{d}_{m_1})(\vec{e}\cdot\vec{d}_{m_2})(\vec{e}\cdot\vec{d}_{m_3})(\vec{e}\cdot\vec{d}_{m_4})]$$
$$-3 \sum_{k,l,m_1,m_2} \frac{\delta(\ldots)}{\sqrt{|k l m_1 m_2|}} \mathcal{I}[(\vec{d}_k\cdot\vec{d}_l)(\vec{e}\cdot\vec{d}_{m_1})(\vec{e}\cdot\vec{d}_{m_2})]$$
$$+\frac{1}{4} \sum_{k_1,k_2,l_1,l_2} \frac{\delta(\ldots)}{\sqrt{|k_1 k_2 l_1 l_2|}} \mathcal{I}[(\vec{d}_{k_1}\cdot\vec{d}_{l_1})(\vec{d}_{k_2}\cdot\vec{d}_{l_2})],$$

where the terms correspond to $p = 0, 1, 2$ respectively. We'll compute each term separately; all the possible 'topologically' different pairings are drawn in figure 2, and we'll refer to them in the equations as $(\quad p = 0 :\ (a)\ )$ etc.

The $p = 0$ term: this is the easiest term and readily gives

$$\begin{aligned} &4\left\{(\ p=0:\ (a)\ )\cdot 3\cdot 2^2 + (\ p=0:\ (b)\ )\cdot 6\right\} \\ &\qquad = 4\left\{12\sum_{\neq}\frac{1}{m_1 m_2} N^a_{m_1} N^b_{m_2} e_a^2 e_b^2 + 6\sum \frac{1}{m^2} N^a_m N^b_m e_a^2 e_b^2\right\} \\ &\qquad = 4\left\{12\Big(\sum \frac{N_m}{m}\Big)^2 - 6\sum\frac{N_m^2}{m^2}\right\} \\ &\qquad = 48 M_1^2 - 24 M_2. \end{aligned} \tag{A.5}$$

This requires some explanation. The sums are over all indices ($\{k_i, l_i, m_j\}$) present and run from 1 to infinity, and $\sum_{\neq}$ is shorthand for $\sum_{m_1 \neq m_2}$. Sums over the $\mathbb{R}^4$ indices are also

[10] The $n = 1$ case, which turns out to vanish, is too simple and does not illustrate the method particularly well.

present, though we've suppressed them. The degeneracies on the first line are as follows: 3 is due to m_1 being able to pair up with any of the three other indices, and 2^2 is due to there being two pairs, in each of which the positive index can be chosen in two ways. In the second term, $6 = \binom{4}{2}$ is the number of ways two of the four indices can be chosen to be positive. Also note that these degeneracies coincide with the number of terms as given earlier, namely $12 = \frac{4!}{2!}$ and $6 = \frac{4!}{2!2!}$. Checking that this is always satisfied is a vital consistency check to make sure the degeneracies are taken into account correctly. We should also point out that from the formalism above it is clear that the answer can always be given as a sum of the multipoles, such that the powers are correct, for instance M_2 or M_1^2 here.

The $p = 1$ term: for the remaining terms, we only give the beginning and the end of the computation; using (A.4) it is straightforward to fill in the missing steps. The $p = 1$ term gives

$$-3\{(\ p=1:\ (a)\ )\cdot 4 + (\ p=1:(b)\ )\cdot 2\cdot 4 + (\ p=1:\ (c)\ )\cdot 6\} = -72M_1^2 + 18M_2, \tag{A.6}$$

where we again check that the degeneracies are correct: $4 + 2\cdot 4 = 12$ and 6, which is correct.

The $p = 2$ term: finally, for $p = 2$ we get

$$\frac{1}{4}\{(\ p=2:\ (a)\ )\cdot 4 + (\ p=2:\ (b)\ )\cdot 2\cdot 4 + (\ p=2:\ (c)\ )\cdot 6\} = 24M_1^2 - 6M_2, \tag{A.7}$$

where again the degeneracies match.

Putting these results together we get that the $\frac{1}{r^6}$ term in the asymptotic expansion of the metric is $-12\frac{Q_5\mu^4}{r^6}M_2$, as given in (3.5). Note that the M_1^2 terms cancel, leaving only M_2. At the $n = 3$ level, one can show that the M_1^3 and M_1M_2 terms cancel, leaving only the M_3 term. It is tempting to conjecture that this cancellation always happens, but we've been unable to show this. Nevertheless, the arguments of this paper are not sensitive to whether terms like M_1^k etc. are present at level k along with the M_k term.

B. Expansion of f_1

For completeness we will also compute the asymptotic form of the f_1 function (2.25). Since f_1 differs from f_5 only by inclusion of an $|\vec{F}'(s)|^2$ term, we can follow the same recipe as for f_5, and we find

$$\begin{aligned} f_1 &= 1 + \frac{Q_5}{r^2}\frac{2\pi^2\mu^2}{L^2}\sum_{n=0}^{\infty}\left(\frac{\mu}{r}\right)^{2n}\sum_{p=0}^{n}(-1)^{p+1}2^{n-2p}\binom{2n-p}{p}\times \\ &\times \sum_{\substack{k_1,\ldots,k_{p+1}\\ l_1,\ldots,l_{p+1}\\ m_1,\ldots,m_{2(n-p)}}}\delta\Big(\sum_i(k_i+l_i)+\sum_j m_j\Big)\frac{k_{p+1}l_{p+1}}{\sqrt{|\prod_i k_il_i\prod_j m_j|}}\mathcal{N}\times \\ &\times\int_{\vec{d}}\prod_{k=1}^{\infty}\prod_{a=1}^{4}e^{-d_k^a d_k^{a*}}\,(d_k^a d_k^{a*})^{N_k^a}\prod_{i=1}^{p+1}\left(\vec{d}_{k_i}\cdot\vec{d}_{l_i}\right)\prod_{j=1}^{2(n-p)}\left(\vec{e}\cdot\vec{d}_{m_j}\right). \end{aligned} \tag{B.1}$$

The difference to the expansion of f_5 is the inclusion of a factor $-\frac{2\pi^2\mu^2}{L^2}k_{p+1}l_{p+1}\vec{d}_{k_{p+1}}\cdot\vec{d}_{l_{p+1}}$. Note that using equations (2.10), (2.12) and (2.13), we can write $Q_5\frac{2\pi^2\mu^2}{L^2} = \frac{Q_1}{2N}$.

The $n = 0$ term: the first term is given by

$$-\frac{Q_1}{r^2}\frac{1}{2N}\sum_{k,l}\delta(k+l)\frac{kl}{\sqrt{|kl|}}\mathcal{I}[(\vec{d}_k\cdot\vec{d}_l)] = \frac{Q_1}{r^2}\frac{1}{2N}\cdot 2\sum_{k=1}^{\infty}\sum_{a=1}^{4}kN_k^a = \frac{Q_1}{r^2}, \tag{B.2}$$

which is of course the expected result.

The $n = 1$ term: at the $n = 1$ level we have two terms: $p = 0, 1$. The first one yields

$$-\frac{Q_1\mu^2}{r^4}\frac{1}{2N}2\sum_{k,l,m_1,m_2}\delta(k+l+m_1+m_2)\frac{kl}{\sqrt{|klm_1m_2|}}\mathcal{I}[(\vec{d}_k\cdot\vec{d}_l)(\vec{e}\cdot\vec{d}_{m_1})(\vec{e}\cdot\vec{d}_{m_2})], \tag{B.3}$$

and we see that the possible pairings are just those from the second row of figure (2). However, the pairing (b) does not contribute in this case; the reason is that if k and l are independent the sums will yield zero as the summand is odd in both k and l. Thus k and l will always have to be linked to produce a contribution. Thus we get

$$\begin{aligned}
-\frac{Q_1\mu^2}{r^4}\frac{1}{N}&((\ (a)\ )\cdot 4 + (\ (c)\ )\cdot 2) \\
&= -\frac{Q_1\mu^2}{r^4}\frac{1}{N}\left(-4\sum_{k\neq m}\frac{k}{m}N_k^aN_m^be_b^2 - 4\sum_{k=1}^{\infty}\frac{k}{k}N_k^aN_k^be_b^2 + 2\sum_{k=1}^{\infty}\frac{k}{k}N_k^aN_k^ae_a^2\right) \\
&= \frac{Q_1\mu^2}{r^4}\frac{1}{N}\left(4\sum_{k,m}\frac{k}{m}N_kN_m - 2\sum_{k=1}^{\infty}\frac{k}{k}N_k^aN_k^ae_a^2\right) \\
&= \frac{Q_1\mu^2}{r^4}\left(4M_1 - \frac{2}{N}\sum_{k=1}^{\infty}N_k^2\right).
\end{aligned} \tag{B.4}$$

For the $p = 1$ term we see that the possible pairings are given on the third line of figure 2, except that (b) again does not contribute, for the same reason as stated above. The computation proceeds as above and after some algebra we get

$$\frac{Q_1\mu^2}{r^4}\frac{1}{2N}\frac{1}{2}((\ (a)\ )\cdot 4 + (\ (c)\ )\cdot 6) = \ldots = \frac{Q_1\mu^2}{r^4}\left(-4M_1 + \frac{2}{N}\sum_{k=1}^{\infty}N_k^2\right). \tag{B.5}$$

Thus we see that the $p = 0$ and $p = 1$ terms cancel, and at level $n = 1$ there is no contribution, which is exactly what happened for the f_5 expansion as well.

The $n = 2$ term: finally, we can also compute the $n = 2$ term. Here the combinatorics are already somewhat complicated, so we won't present the computation. However, in the end we can write the expansion to order $\frac{1}{r^6}$ as

$$f_1 = 1 + \frac{Q_1}{r^2} - \frac{Q_1\mu^4}{r^6}\left(12M_2 - \frac{16}{N}\sum_{k=1}^{\infty}\frac{N_k^3}{k}\right) + \mathcal{O}(\frac{1}{r^8}). \tag{B.6}$$

Again we see that terms with M_1 have cancelled, leaving only M_2. However, now we also have a term of the form $\sum_{k=1}^{\infty} N_k^3/k$, which is not one of the multipoles we have defined. Furthermore, from the formalism we see that the new objects that can appear are of the form $\sum_k N_k^n/k^{n-2}$, where the mismatch in powers is due to the factor $k_{p+1}l_{p+1}$ that came from including $|\vec{F}'(s)|^2$. In principle we should make sure that these quantities don't fluctuate too much either, but due to their great similarity to the multipoles, it is clear that if we fix the multipoles with accuracies ϵ_k, then these new objects will also be fixed by some set of frequencies ϵ'_k. Thus we shall not worry about these objects in this paper.

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